

ON THE EXISTENCE OF PRIMITIVE PENCILS FOR SMOOTH CURVES

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ABSTRACT. Let C be a smooth curve with gonality $k \geq 6$ and genus $g \geq 2k^2 + 5k - 6$. We prove that $W_d^1(C)$ has the expected dimension and that the general element of any irreducible component of $W_d^1(C)$ is primitive if either $g - k + 4 \leq d \leq g - 2$ or $d = g - k + 3$ and either k is odd or C is not a double covering of a curve of gonality $k/2$ and genus $k - 3$. Even in the latter case we prove the existence of a complete and primitive g_{g-k+3}^1 .

A line bundle L on a smooth curve C of genus $g \geq 4$ is said to be *primitive* if it is spanned and both L and $\omega_C \otimes L^\vee$ are spanned, i.e. if it is spanned and $h^0(L(q)) = h^0(L)$ for all $q \in C$ (sometimes one also imposes that $L \neq \mathcal{O}_C$ and $L \neq \omega_C$) ([7], [8]). Since L is primitive if and only if $\omega_C \otimes L^\vee$ is primitive, to study primitive line bundles on C it is sufficient to know the ones with $0 < \deg(L) \leq g - 1$. Let C be a smooth curves of genus $g \geq 4$ with gonality k . If either $g \neq k(k-1)/2$ or C is not isomorphic to a smooth plane curve, then C has a complete and primitive g_k^1 . For very low k or for a general smooth curve of genus g the Brill-Noether theory of C is well-known and it gives a complete description of the complete and primitive g_d^r on C ([7], [8], [9], [10]). If $g \geq 11$ and $k \geq 5$, then a general element of any irreducible component of $W_{g-2}^1(C)$ is primitive ([18, Proposition II.0]). In this note we consider the existence of complete and primitive g_d^1 for all d near $g - 1$ and prove the following result.

Theorem 1. *Fix an integer $k \geq 6$ and set $g(k) := 2k^2 + 5k - 6$. Fix any integer $g \geq g(k)$, any smooth curve C with gonality k and any integer d with $g - k + 3 \leq d \leq g - 2$.*

(a) *C has a complete and primitive g_d^1 and every irreducible component of $W_d^1(C)$ has dimension $2d - g - 2$.*

(b) *Assume that either k is odd or $d > g - k + 3$ or $d = g - k + 3$, k is even, but C is not a double covering of a smooth curve of genus $k - 3$ and gonality $k/2$. Then a general element of every irreducible component of $W_d^1(C)$ is primitive.*

(c) *Assume that k is even and that C is a double covering of a smooth curve of genus $k - 3$ and gonality $k/2$.*

(c1) *There exist an irreducible component of $W_{g-k+3}^1(C)$ whose general member is base point free and an irreducible component of $W_{g-k+3}^1(C)$ whose general member has $g - 2k + 3$ base points.*

(c2) *Let Γ be any irreducible component of $W_{g-k+3}^1(C)$; if the general element of Γ is base point free, then it is primitive.*

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Remark 1. Fix an integer $x \geq 3$. Set $w := \lfloor x/2 \rfloor$, $z := \lfloor (x-1)/2 \rfloor$, $g_1(x) := 2x(2x-z) - 4x + z + 2$ and $g_2(x) := 4x^2 - 2wx + w$. Fix an integer $g \geq \max\{g_1(x), g_2(x)\}$. Let C be a smooth curve of genus g and gonality at least $x+3$. The interested reader may reformulate an analogous of Theorem 1 with $d = g - x$ and prove it following verbatim the proof of Theorem 1. In the case $g - d = 4$ of Theorem 1 it is sufficient to assume that $g \geq 64$ (see Theorem 2).

Remark 2. In the exceptional cases of Theorem 1 we have a description of the irreducible components of $W_{g-k+3}^1(C)$ whose general element has base points. We have $\dim(W_k^1(C)) = 1$ and each element of these components is obtained from some $R \in W_k^1(C)$ adding a base locus of degree $g - 2k + 3$. Let Y be any smooth curve of genus g and gonality k with $\dim(W_k^1(Y)) = 1$. If $W_{g-k+3}^1(Y)$ has pure dimension $g - 2k + 4$, then it has at least one component formed by pencils with a base locus of degree $g - 2k + 3$. Steps (a) and (b) of the proof of Theorems 1 show that if $g \gg k$, then k is even and Y is a double covering of a smooth curve of genus $k - 3$ and gonality $k/2$.

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1. THE PROOFS

Lemma 1. Fix integers g, x such that $x \geq 2$ and $g \geq 4x + 3$. Let C be a smooth curve of genus g . Let T be an irreducible component of $W_{g-x}^1(C)$.

- (a) If $\dim(T) > g - 2x - 2$, then $\dim(W_{2x}^1(C)) \geq x - 1$.
- (b) If a general element of T has a base point, then $\dim(W_{2x+2}^1(C)) \geq x$.

Proof. Since $g \geq 2x + 2$, Brill-Noether theory gives $W_{g-x}^1(C) \neq \emptyset$ and that each irreducible component of $W_{g-x}^1(C)$ has dimension at least $g - 2x - 2$ ([1, Ch. IV]).

First assume $\dim(T) \geq g - 2x - 1$. Set $d := g - x$ and $j = x - 1$. We have $g - 2x - 1 = d - 2 - j$, $d \geq j + 2$ and $d \leq g - 1 - j$ (the latter inequality is an equality). By [17] or [12, Theorem 1] we have $\dim(W_{2x}^1(C)) \geq x - 1$.

Now assume that a general element of T has at least one base point. We get an irreducible component of $W_{g-x-1}^1(C)$ with dimension at least $g - 2x - 3$. Apply part (a) with the integer $x' := x + 1$ instead of the integer x . \square

Proof of Theorem 1: Since $2d - g - 2 \geq 0$, Brill-Noether theory says that $W_d^1(C) \neq \emptyset$ and that each irreducible component of $W_d^1(C)$ has dimension at least $2d - g - 2$. Let T be an irreducible component of $W_d^1(C)$ and let R be a general element of T . As in the case of the general member of any irreducible component of any $W_y^1(C)$ with $y \leq g - 1$ we have $h^0(R) = 2$. To prove Theorem 1 it is sufficient to prove that $\dim(T) = 2d - g - 2$, that R is base point free and that $h^0(R(p)) = 1$ for all $p \in C$. Let $f : X \rightarrow \mathbb{P}^1$ be any degree k morphism.

(a) Assume $\dim(T) > 2d - g - 2$. The case $x = g - d$ of part (a) of Lemma 1 gives $\dim(W_{2g-2d}^1(C)) \geq g - d - 1$. Let Γ be any irreducible component of $W_{2g-2d}^1(C)$ with $\dim(\Gamma) \geq g - d - 1$. Let R' be a general element of Γ . Since R' is general in an irreducible component of some $W_y^1(C)$, $y \leq g + 1$, we have $h^0(R') = 2$. Let $s \geq 0$ be the degree of the base locus B of R' . Varying R' we get an irreducible family $\Gamma' \subseteq W_{2g-2d-s}^1(C)$ with $\dim(\Gamma') \geq g - d - 1$. Set $R'' := R'(-B)$. Let $u : C \rightarrow \mathbb{P}^1$ be the morphism associated to $|R''|$ and let $\alpha : C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the morphism associated to (f, u) . If α is birational onto its

image, then $g \leq k(2g-2d-s)-k-(2g-2d-s)+1 \leq k(2k+6)-k-(2k+6)+1 < g(k)$, a contradiction. Hence C is not birational onto its image, i.e. calling C' the normalization of $\alpha(C)$ we get a morphism $\beta' : C \rightarrow C'$ with $\beta := \deg(\beta') \geq 2$ and morphisms $f' : C' \rightarrow \mathbb{P}^1$ and $u' : C' \rightarrow \mathbb{P}^1$ such that $f = f' \circ \beta'$ and $u = u' \circ \beta'$. Since f computes the gonality of C , we get that C' has genus $q_{R'} > 0$ and that C' has gonality k/β .

First assume $q_{R'} \geq 2$. Since Γ is irreducible and C has only finitely many non-constant morphisms to curves of genus between 2 and $g-1$ by a theorem of de Franchis, we get that C' , β and β' do not depend from the choice of R' . Since $h^0(R'') = 2$, we have $h^0(C, L'') = 2$, where L'' is the line bundle on C' with $\beta'^*(L'') \cong R''$. Since $h^0(R'') = 2$, we have $h^0(L'') = 2$ and hence $(2g-2d-s)/\beta \leq q+1$. By [11, Theorem 1] we have $W_{(2g-2d-s)-\beta[(g-d-1)/2]}^1(C) \neq \emptyset$ and hence $k \leq (2g-2d-s) - \beta[(g-d-1)/2] \leq (2g-2) - \beta[(g-d-1)/2]$. Since $\beta \geq 2$ and $g-d \leq k-3$, we get $k \leq 2g-2d-g+d+2 \leq 2k-6-k+5$, a contradiction.

Now assume $q_{R'} = 1$. Since C' has gonality k/β , we get $\beta = k/2$. Hence $2g-2d-s$ is divisible by $k/2$. Since $d \leq g-2$, we have $\dim(W_{2g-2d-s}^1(C)) \geq 3$ and hence $2g-2d-s > k$. Therefore $2g-2d-s \geq 3k/2$. Since $q_{R'} = 1$, we have $h^0(C', L'') = \deg(L'') \geq 3 > h^0(R'')$, a contradiction.

(b) In this step we prove that R has no base points if one of the conditions in part (b) of the statement of Theorem 1 is satisfied. If R has a base point, then part (b) of Lemma 1 with $x = g-d$ gives $\dim(W_{2g-2d+2}^1(C)) \geq g-d$. Let Γ_1 be any irreducible component of $W_{2g-2d+2}^1(C)$ with $\dim(\Gamma_1) \geq g-d$. Let R' be a general element of Γ_1 . Since R' is general in an irreducible component of some $W_y^1(C)$, $y \leq g+1$, we have $h^0(R') = 2$. Let $s \geq 0$ be the degree of the base locus B of R' . Varying $R'' := R'(-B)$ we get an irreducible family $\Gamma' \subseteq W_{2g-2d+2-s}^1(C)$ with $\dim(\Gamma') \geq g-d$ and with R'' as its general member. Let $u : C \rightarrow \mathbb{P}^1$ be the morphism associated to $|R''|$ and let $\alpha : C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the morphism associated to (f, u) . If α is birational onto its image, then $g \leq k(2g-2d+2-s)-k-(2g-2d+2-s)+1 \leq k(2k+8)-k-(2k+8)+1 = g(k)-1$, a contradiction. Hence C is not birational onto its image, i.e. calling C' the normalization of $\alpha(C)$ we get a morphism $\beta' : C \rightarrow C'$ with $\beta := \deg(\beta') \geq 2$ and morphisms $f' : C' \rightarrow \mathbb{P}^1$ and $u' : C' \rightarrow \mathbb{P}^1$ such that $f = f' \circ \beta'$ and $u = u' \circ \beta'$. Since f computes the gonality of C , we get that C' has genus $q_{R'} > 0$ and that C' has gonality k/β .

First assume $q_{R'} \geq 2$. Since Γ is irreducible and C has only finitely many non-constant morphisms to curves of genus between 2 and $g-1$ by a theorem of de Franchis, we get that C' , β' and β does not depend from the choice of R' . Since $h^0(R'') = 2$, we have $h^0(C, L'') = 2$, where L'' is the line bundle on C' with $\beta'^*(L'') \cong R''$. Since $h^0(R'') = 2$, we have $h^0(L'') = 2$ and hence $(2g-2d+2-s)/\beta \leq q+1$. By [11, Theorem 1] we have $W_{2g-2d+2-2s-\deg(B_1)-\beta[(g-d)/2]}^1(C) \neq \emptyset$ and hence $k \leq (2g-2d+2-s) - \beta[(g-d)/2]$. Since $2 \leq g-d \leq k-3$, we get $k = g-d+3$, $\beta = 2$, $s = 0$, $g-d$ odd and $q \geq k-3$. We also get that C' has gonality $k/2$. Since $\dim(W_{k-2}^1(C')) \geq \dim(\Gamma_1) \geq k-3 \geq q$, we get $q = k-3$. We are in the exceptional case allowed in the statement of Theorem 1.

Now assume $q_{R'} = 1$. Since C' has gonality k/β , we get $\beta = k/2$. Hence $2g-2d-s$ is divisible by $k/2$. Since $d \leq g-2$, we have $\dim(W_{2g-2d-s}^1(C)) \geq 3$ and hence $2g-2d-s > k$. Therefore $2g-2d-s = 3k/2$. Since $q_{R'} = 1$, we have $h^0(C', L'') = \deg(L'') = 3 > h^0(R'')$, a contradiction.

(c) Assume the existence of $p_R \in C$ such that $h^0(R(p_R)) = 3$. Since R is base point free and $h^0(R) = 2$, $M := R(p_R)$ is base point free. Let $u : C \rightarrow \mathbb{P}^2$ be the morphism induced by $|M|$. Since R is general in T , we get $\dim(W_{d+1}^2(C)) \geq 2d - g - 3$ and $\dim(W_{d+1}^2(C)) \geq 2d - g - 2$, unless the same general M comes from infinitely many pairs (R_1, P_{R_1}) with R_1 general in T . Since a smooth plane curve of degree $d + 1$ has gonality $d > k$, either $\deg(u) > 1$ or $u(C)$ is a singular curve. First assume $\deg(u) = 1$. Taking the a linear projection from one of the finitely many singular points of $u(C)$ we get $\dim(W_{d-1}^1(C)) \geq 2d - g - 3$. The case $x_1 := x + 1$ of part (a) of Lemma 1 gives $\dim(W_{2g-2d+2}^1(C)) \geq g - d$. We are in the exceptional case described in step (b). Now assume $\deg(u) > 1$. If $\dim(W_{d+1}^2(C)) \geq 2d - g - 2$, we get the same lower bound for $\dim(W_{d-1}^1(C))$ taking a linear projection from any point of $u(C)$. Taking a linear projection we get a better estimate if either $\dim(W_{d+1}^2(C)) \leq 2d - g - 3$ or $\deg(u) \geq 3$ or $u(C)$ is singular. Now assume $\deg(u) = 2$ and that $u(C)$ is smooth. Since $\deg(u(C)) = (d + 1)/2 \geq (g - k + 34)/2$, we get that $u(C)$ has genus $q' \geq (g - k + 3)(g - k + 2)/8$. Since $g \geq 2q' - 1$ (Riemann-Hurwitz), we get a contradiction.

(d) To conclude the proof we may assume that k is even and the existence of a degree 2 covering $\beta' : C \rightarrow C'$ with C' smooth of genus $k - 3$ and gonality $k/2$. By step (a) $W_{g-k+3}^1(C)$ has pure dimension $g - 2k + 4$. Remark 2 gives the existence of an irreducible component of $W_{g-k+3}^1(C)$ whose general member has $g - 2k + 3$ base points. Brill-Noether theory gives $\dim(W_t^1(C')) \geq 2t - k + 1$ for all $t \in \mathbb{N}$ such that $k/2 \leq t \leq k - 2$. Since C' has gonality $k/2$, [11, Theorem 1] first gives $\dim(W_{k/2}^1(C')) = 1$ and then $\dim(W_t^1(C')) \leq 2t - k + 1$ for all $t \in \mathbb{N}$ such that $k/2 < t \leq k - 2$. Hence C' has dimensionally the Brill-Noether theory for pencils of a general curve of genus $k - 3$. This is enough to carry over the proof of [2, Theorem 0.1] (see the proofs of Lemmas 1.2, 1.3 and Theorem 0.1 in [2]). Hence (with this observation concerning C'), [2, Theorem 0.1] gives the existence of a degree $g - k + 3$ morphism $f : C \rightarrow \mathbb{P}^1$ not composed with β' , i.e. such that the morphism $(\beta', f) : C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is birational onto its image. We claim that we may take as f a complete pencil. This claim is true by [2, Lemma 1.3], which describes all the $W_{g-k+3}^1(C)$, $k - 3 = p_a(C')$, whose general element has base points and the existence of at least another components of $W_{g-k+3}^1(C)$ ([2, first line of page 155]; it is the second line of page 155, which loses the completeness statement in [2, Theorem 0.1]). So we proved the existence of an irreducible component of $W_{g-k+3}^1(C)$ containing a base point free and complete g_{g-k+3}^1 .

Let Γ be any irreducible component of $W_{g-k+3}^1(C)$ containing a base point free and complete g_{g-k+3}^1 , δ . Since δ is complete, the general element of Γ is complete and base point free. Let R be a general element of Γ . Assume the existence of $p_R \in C$ such that $h^0(R(p_R)) = 3$. Since R is base point free and $h^0(R) = 2$, $M := R(p_R)$ is base point free. Since R is general in Γ , we get $\dim(W_{g-k+4}^2(C)) \geq g - 2k + 3$ and $\dim(W_{g-k+4}^2(C)) \geq g - 2k + 4$, unless a general M comes from infinitely many pairs (R, p_R) . Let $u : C \rightarrow \mathbb{P}^2$ be the morphism induced by $|M|$. First assume $\deg(u) = 1$. Since $g - k + 4 > k + 1$ and a smooth plane curve of degree $g - k + 4$ has gonality $g - k + 3 > k$, $u(C)$ is a singular curve. Therefore taking a linear projection from a singular point of $u(C)$ we obtain $\dim(W_{g-k+2}^1(C)) \geq g - 2k + 3$ (because $u(C)$ has only finitely many singular points). Write $\dim(W_{g-k+2}^1(C)) = g - k - 1 - j + e$ with $e \geq 0$ and $j = k - 4$. By [12, Theorem 1] we have $\dim(W_{2k-6-2e}^1(C)) = k - 1 - e$

and (hence, even if $e > 0$ by [11, Theorem 1]) we have $\dim(W_{2k-6}^1(C)) \geq k - 4$. This is the case handled in step (a). Now assume $\deg(u) > 1$ and call C'' the normalization of $u(C)$, q its genus, and $v : C \rightarrow C''$ the morphism through which u factors. Write $M = v^*(L)$ with L base point free line bundle on C'' with $h^0(C, L) = 3$. If $\deg(u) \geq 3$, then fixing $o \in u(C)_{\text{reg}}$ and taking the linear projection from o we get $\dim(W_{g-k+1}^1(C)) \geq g - 2k + 3$ (the same for all M , because $q \geq 2$ and we may apply a theorem of de Franchis). Write $\dim(W_{g-k+1}^1(C)) = g - k - 1 - j + e'$ with $e' \geq 0$ and $j = k - 4$. We get $\dim(W_{2k-8}^1(C)) \geq k - 3$ and conclude. Now assume $\deg(u) = 2$. In this case $(g - k + 4)/2 \in \mathbb{Z}$. If $u(C)$ is singular, then a linear projection from one of its singular points gives $\dim(W_{g-k}^1(C)) \geq g - 2k + 3$ and so $\dim(W_{g-k+3}^1(C)) \geq g - 2k + 6$, contradicting step (a). Hence $u(C)$ is smooth and so it has genus $q' := (g - k + 2)(g - k)/8$. Riemann-Hurwitz gives $g \geq 2q' - 1$, a contradiction. \square

Example 1. Fix an even integer $k \geq 6$ and set $x := k - 3$. Let C' be any smooth curve of genus x and gonality $k/2$. We have $\dim(W_{k/2}^1(C')) = 1$. Let $u : C \rightarrow C'$ be a degree 2 covering of genus $g \geq 3x + 4$. $W_{g-x}^1(C)$ contains the $(g - 2x - 2)$ -dimensional family of g_{g-x}^1 obtained from the pull-backs of the elements of $W_{k/2}^1(C')$ adding $g - 2x - 3$ base points. Since $g > 2x + k$, any base point free pencil on C of degree $< k$ is the pull-back of a base point free pencil on C' by the Castelnuovo-Severi inequality ([13]). Hence C has gonality k . Hence the exceptional cases in Theorem 1 arises.

See also [4] (resp. [5]) for the existence of spanned pencils on curves which are double (resp. multiple) coverings. By [3, Theorem 2.2] for every integer $d \geq g - k + 2$ every k -gonal curves of genus $g > (3k - 6)(k - 1)$ has a base point free g_d^1 .

Theorem 2. *Let C be a smooth curve of genus $g \geq 64$ with gonality $k \geq 7$. Then C has a primitive g_{g-4}^1 , $W_{g-4}^1(C)$ has pure dimension $g - 10$ and the general element of every irreducible component of $W_{g-4}^1(C)$ is primitive.*

Proof. Since $2(g - 4) - g - 2 \geq 0$, Brill-Noether theory gives $W_{g-4}^1(C) \neq \emptyset$ and that any irreducible component T of $W_{g-4}^1(C)$ has dimension at least $2(g - 4) - g - 2 = g - 10$. Fix a general $R \in T$. As in the case of any irreducible component of any $W_d^r(C)$ we have $\dim |R| = 1$. To prove Theorem 2 it is sufficient to prove that R is base point free, that $h^0(R(q)) = 2$ for every $q \in C$, and that $\dim T = g - 10$.

(a) In this step we prove that R has no base points. If R has a base point, then the case $x = 4$ of part (b) of Lemma 1 gives $\dim(W_{10}^1(C)) \geq 4$.

(a1) Assume $k = 7$. Let $h : C \rightarrow \mathbb{P}^1$ be any degree 7 morphism. Since 7 is a prime number and $g > 7 \cdot 7 - 7 - 7 + 1$, the genus formula for integral curves contained in $\mathbb{P}^1 \times \mathbb{P}^1$ shows that h is unique. Let m be the first integer > 7 such that C has a base point free g_m^1 . Since $\dim(W_{10}^1(C)) > 10 - 7$ and $\dim(W_7^1(C)) = 0$, we have $m \leq 10$. Every integral curve of $\mathbb{P}^1 \times \mathbb{P}^1$ with bidegree $(7, m)$ has arithmetic genus $7m - 7 - m + 1 \leq 70 - 7 - m + 1 < g$, a contradiction.

(a2) Assume $k = 8$. First assume that C has only finitely many g_8^1 . Let $h : C \rightarrow \mathbb{P}^1$ be any degree 8 morphism. Since $\dim(W_9^1(C)) \geq 2$, we get the existence of a degree 9 morphism $f : C \rightarrow \mathbb{P}^1$. Since 8 and 9 are coprime, the map $(h, f) : C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is birational onto its image, Γ . Since $p_a(\Gamma) = 8 \cdot 9 - 8 - 9 + 1$, we get $g \leq 56$, a contradiction. Now assume $\dim(W_8^1(C)) \geq 1$ and hence $\dim(W_8^1(C)) = 1$. Since $g > 8 \cdot 9 - 8 - 9 + 1$, the proof of step (a1) shows that every element of $W_9^1(C)$

has one base point. Since $\dim(W_{10}^1(C)) > 3 + \dim(W_8^1(C))$, there is a degree 10 morphism $\ell : C \rightarrow \mathbb{P}^1$. Since $g > 8 \cdot 10 - 8 - 10 + 1$, we get that $(h, \ell) : C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ has degree 2 onto its image $\Phi_{(h, \ell)}$ and $\Phi_{(h, \ell)}$ is an integral curve of bidegree $(4, 5)$. Since $k > 4$, the normalization $D_{(h, \ell)}$ of $\Phi_{(h, \ell)}$ has genus > 1 . Hence varying u and ℓ we only get finitely many smooth curves $D_{(h, \ell)}$. We fix one such normalization $D = D_{(h, \ell)}$ and call $v : C \rightarrow D$ the map induced by (u, ℓ) and q the genus of D . We have $q \leq 4 \cdot 5 - 4 - 5 + 1 = 12$ by the Castelnuovo - Severi inequality. Since $g > 2q + 9$ all base point free pencils of degree 8 (resp. 10) pencils on C are induced by a degree 4 (resp. 5) pencil on D (again by the Castelnuovo - Severi inequality). Since $\dim(W_{10}^1(C)) \geq 4$, we get $\dim(W_5^1(D)) \geq 4$. Hence $W_3^1(C) \neq \emptyset$ ([11]). Therefore $k \leq 6$, a contradiction.

(b) Assume $\dim(T) > g - 10$. The case $x = 4$ of part (a) of Lemma 1 gives $\dim(W_8^1(C)) \geq 3$. Hence $k \leq 7$ and if $k = 7$, then $\dim(W_7^1(C)) = 1$. Assume $k = 7$. Since 7 is a prime number and C has at least two g_7^1 , the Castelnuovo - Severi inequality gives $g \leq 7 \cdot 7 - 7 - 7 + 1$, a contradiction.

(c) Assume the existence of $q_R \in C$ such that $h^0(R(q_R)) = 3$. Since R is general in T , we get $\dim(W_{g-3}^2(C)) \geq g - 11$. Since R is base point free and $h^0(R) = 2$, $M := R(q_R)$ is base point free. Let $u : C \rightarrow \mathbb{P}^2$ be the morphism induced by $|M|$. Since $g < (g-4)(g-5)/2$ either $\deg(u) > 1$ or $u(C)$ is a singular curve. Therefore taking a linear projection from a point of $u(C)$ (case $\deg(u) > 1$) or a singular point of $u(C)$ (case $\deg(u) = 1$), we obtain $\dim(W_{g-5}^1(C)) \geq g - 11$. The case $x = 5$ of part (a) of Lemma 1 gives $\dim(W_{10}^1(C)) \geq 4$. We are in the case excluded in step (a). \square

Remark 3. Let C be a smooth curve of genus g with a primitive $R \in \text{Pic}^d(C)$, $d \leq g - 2$. Since $\omega_C \otimes R^\vee$ is primitive, C has a primitive g_{2g-2-d}^{g-d} . In the case $k \geq 5$ and $d = g - 2$ for a general R the dual linear series $\omega_C \otimes R^\vee$ is birational onto its image and with image a plane nodal curve ([18, Proposition II.0]). See [14], [15], [9] for the very ampleness of some g_{g+1}^3 (case $d = g - 3$).

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